Warwick Mathematics Exchange

MA260

Norms, Metrics and Topologies

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Desync, aka The Big Ree

Contents

Introduction

In *Norms, Metrics and Topologies*, we generalise the notion of lengths and distances with normed and metric spaces, before investigating topological properties of those spaces. This document is intended to broadly cover all the topics within the Norms, Metrics and Topologies module.

Disclaimer: I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. This document was written during the 2022 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be underlined. The latter two classifications are under my interpretation. YMMV.

Content not taught in the course will be outlined in the margins like this. Anything outlined like this is not examinable, but has been included as it may be helpful to know alternative methods to solve problems.

The table of contents above, and any inline references are all hyperlinked for your convenience.

Scalars are written in lowercase italics, c, or using Greek letters.

Vectors are written in lowercase bold, $\mathbf{v}$, or rarely overlined, $\overleftrightarrow{v}$, where more contrast or clarity is required.

Matrices are written in uppercase bold, $\mathbf{A}$.

Note: transformations represented by matrices may be written in just italics, as functions often are, i.e., $s(\mathbf{v}) = \mathbf{A}\mathbf{v}$.

History

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This document was written by R.J. Kit L., a maths student. I am not otherwise affiliated with the university, and cannot help you with related matters.

Please send me a PM on Discord @Desync#6290, a message in the WMX server, or an email to Warwick.Mathematics.Exchange@gmail.com for any corrections. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support me, you can buy me a coffee!

(Direct link for if hyperlinks are not supported on your device/reader: ko-fi.com/desync.)

*Storing dates in big-endian format is clearly the superior option, as sorting dates lexicographically will also sort dates chronologically, which is a property that little and middle-endian date formats do not share. See ISO-8601 for more details. This footnote was made by the computer science gang.

1 Normed Spaces

A *norm* on a real or complex vector space X is a map $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$ such that,

(i) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$ (point separating or positive-definiteness);

(ii) $\|\lambda\mathbf{x}\| = \lambda\|\mathbf{x}\|$ for all $\lambda \in \mathbb{R}$ or $\mathbb{C}$ and all $\mathbf{x} \in X$ (absolute homogeneity);

(iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality).

Note that these axioms imply that $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in X$. The pair $(X,\|\cdot\|)$ is then called a *normed space.*

As an example, the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a norm on the one-dimensional vector spaces $\mathbb{R}$ and $\mathbb{C}$.

For a vector $\mathbf{x} = (x_1,x_2,\dots,x_n)$ in the vector space $\mathbb{R}^n$, we define the *Euclidean* or *standard* norm as,

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

We also have the *taxicab* or *Manhattan* norm,

$$\|\mathbf{x}\|_{\ell^1} = \sum_{i=1}^{n} |x_i|$$

and the *uniform* or *maximum* norm,

$$\|\mathbf{x}\|_{\ell^\infty} = \max_{1\leq i\leq n} \|x_i\|$$

The *closed unit ball* denoted $\overline{\mathbb{B}}$ or $\mathfrak{B}$ in the normed space $(X,\|\cdot\|)$ is the set,

$$\mathfrak{B}_X = \{\mathbf{x} \in X : \|\mathbf{x}\| \leq 1\}$$

The *open unit ball* denoted $\mathbb{B}$ or B in the normed space $(X,\|\cdot\|)$ is the set,

$$B_X = \{\mathbf{x} \in X : \|\mathbf{x}\| < 1\}$$

Let X be a vector space. A subset $K \subseteq X$ is *convex* if, whenever $\mathbf{x},\mathbf{y} \in K$, then $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in K$ for $0 \leq \lambda \leq 1$. Informally, a set is convex if the straight line segment connecting any two points in the set is contained within the set.

Lemma (Convexity of Balls)**.** *In any normed space $(X,\|\cdot\|)$, the open and closed unit balls are convex.*

Proof. We show the case for the closed ball. The proof for the open ball is analogous.

Let $\mathbf{x},\mathbf{y} \in \mathfrak{B}_X$, then $\|\mathbf{x}\| \leq 1$ and $\|\mathbf{y}\| \leq 1$. Then, for $0 \leq \lambda \leq 1$,

$$\begin{aligned} \|\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\| &\leq |\lambda|\|\mathbf{x}\| + |1-\lambda|\|\mathbf{y}\| && \text{[Triangle Inequality]} \\ &\leq \lambda + (1-\lambda) \\ &= 1 \end{aligned}$$

so $\|\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\| \leq 1$, giving $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in \mathfrak{B}_X$, as required. ■

Lemma (Equivalence of Convexity and Triangle Inequality). *Suppose a function $N : X \to \mathbb{R}_{\geq 0}$ satisfies requirements (i) and (ii) of a norm, and in addition, that the set $\mathfrak{B} \coloneqq \{\mathbf{x} \in X : N(\mathbf{x}) \leq 1\}$ is convex. Then, N satisfies the triangle inequality,*

$$N(\mathbf{x}+\mathbf{y}) \leq N(\mathbf{x}) + N(\mathbf{y})$$

and thus defines a norm on X.

Because verifying the triangle inequality can be quite difficult, this lemma provides a simple way to check if a function defines a norm based on verifying convexity of the closed unit ball the function would generate.

For $p \in [1,\infty]$, the ℓ^p norms on $\mathbb{R}^n$ are defined by,

$$\|\mathbf{x}\|_{\ell^p} \coloneqq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

The standard norm corresponds to the choice $p = 2$, the taxicab norm to $p = 1$, and the max norm to $p = \infty$.

Theorem (Minkowski's Inequality in $\mathbb{R}^n$). *For all $1 \leq p \leq \infty$, if $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, then,*

$$\|\mathbf{x}+\mathbf{y}\|_{\ell^p} \leq \|\mathbf{x}\|_{\ell^p} + \|\mathbf{y}\|_{\ell^p}$$

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are *equivalent* if there exists constants $0 < c_1 \leq c_2$ such that,

$$c_1\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2 \leq c_2\|\mathbf{x}\|_1$$

for every $\mathbf{x} \in X$, or equivalently, if,

$$c_1\mathfrak{B}_{(X,\|\cdot\|_1)} \subseteq \mathfrak{B}_{(X,\|\cdot\|_2)} \subseteq c_2\mathfrak{B}_{(X,\|\cdot\|_1)}$$

This notion of equivalence forms an equivalence relation on the space of norms of X.

Theorem (Equivalence of Finite Norms). *All norms on a finite-dimensional vector space are equivalent.*

The *sequence space* ℓ^p, $1 \leq p \leq \infty$, consists of all sequences $(\mathbf{x}_i)_{i=1}^{\infty}$ such that,

$$\sum_{i=1}^{\infty} |\mathbf{x}_i|^p < \infty$$

(in the case of $p = \infty$, ℓ^∞ is the space of bounded sequences) equipped with the corresponding ℓ^p norm.

The ℓ^p spaces are infinite-dimensional, with the standard basis being given by $(\mathbf{e}_i)_{i=1}^{\infty}$, where,

$$\mathbf{e}_i = (0, 0, \ldots, \underbrace{1}_{i\text{th place}}, 0, \ldots)$$

Note that for any $1 \leq q < p \leq \infty$, there are elements of ℓ^p that are not elements of ℓ^q – for instance, the sequence $(i^{-\frac{1}{q}})_{i=1}^{\infty}$ – so the ℓ^p spaces are nested within each other, with ℓ^∞ being the largest, and ℓ^1 the smallest.

Theorem (Minkowski's Inequality in ℓ^p). *For all $1 \leq p \leq \infty$, if $x,y \in \ell^p$, then $x + y \in \ell^p$ and,*

$$\|x+y\|_{\ell^p} \leq \|x\|_{\ell^p} + \|y\|_{\ell^p}$$

If $(X,\|\cdot\|)$ is a normed space, and Y is a subspace of X, then $(Y,\|\cdot\|)$ is another normed space. Strictly speaking, the norm defined on Y is the restriction of the norm on X to Y, but we denote them by the same symbol as the implied meaning is clear.

We denote by $C([a,b])$ the space of (real-valued) continuous functions defined on the interval $[a,b]$. The usual norm to use on this space is the supremum norm,

$$\|f\|_\infty \coloneqq \sup_{x\in[a,b]} |f(x)|$$

but by the extreme value theorem, this is equivalent to,

$$\max_{x\in[a,b]} |f(x)|$$

The L^p norms are defined on this space analogously to how the ℓ^p norms are defined on $\mathbb{R}^n$:

$$\|f\|_{L^p} \coloneqq \left(\int_a^b |f(x)|^p\right)^{\frac{1}{p}}$$

2 Metric Spaces

In many situations, we care less about a notion of length than about a generalised notion of distance. This generalisation is given in the form of a metric.

Let X be any set. A *metric* d on X is a map $d : X \times X \to \mathbb{R}_{\geq 0}$ such that,

(i) $d(x,y) = 0$ if and only if $x = y$ (point separating or positive-definiteness);

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$ (symmetry);

(iii) $d(a,b) \leq d(a,x) + d(x,b)$ for every $a,b,x \in X$ (triangle inequality).

Note that these axioms imply that $d(x,y) \geq 0$ for all $x,y \in X$. The pair (X,d) is then called a *metric space.*

Theorem (Induced Metric)**.** *If $(X,\|\cdot\|)$ is a normed space, then $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ defines a metric on X.*

Proof. We verify the metric axioms:

(i) If $\mathbf{x} = \mathbf{y}$, then $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \|\mathbf{0}\| = 0$; if $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = 0$, then $\mathbf{x} = \mathbf{y}$;

(ii) $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \|(-1)(\mathbf{y} - \mathbf{x})\| = |-1|\|\mathbf{y} - \mathbf{x}\| = \|\mathbf{y} - \mathbf{x}\| = d(\mathbf{y},\mathbf{x})$;

(iii) $d(\mathbf{a},\mathbf{b}) = \|\mathbf{a} - \mathbf{b}\| \leq \|\mathbf{a} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{b}\| = d(\mathbf{a},\mathbf{x}) + d(\mathbf{x},\mathbf{b})$. ■

The *Euclidean* or *standard* metric on $\mathbb{R}^n$ is the metric induced by the Euclidean ℓ^2 norm:

$$\begin{aligned} d_2(\mathbf{x},\mathbf{y}) &= \|\mathbf{x} - \mathbf{y}\|_{\ell^2} \\ &= \sqrt{\sum_{i=1}^n |x_i - y_i|^2} \end{aligned}$$

The *discrete metric* on any non-empty set X is defined as,

$$d(x,y) \coloneqq \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Every point in a discrete metric space is equally distanced from every other distinct point. The discrete metric is useful for counterexamples, as it is very different from metrics that arise from norms.

Let L be the set of words of length n constructed from a finite alphabet Σ of characters. The *Hamming distance* on L is defined as the number of places in the strings which disagree. For example, the strings $abcdef$ and $aacdef$ have a Hamming distance of 1. This metric has important applications in (en)coding and information theory, as it measures, among other things, the error or noise in a signal.

Let G be a graph. The *graph metric* defined on the vertex set of G is the number of edges in a shortest path connecting two vertices.

The *jungle river metric* on $\mathbb{R}^2$ is defined as,

$$d\big((x_1,y_1)(x_2,y_2)\big) = \begin{cases} |y_1 - y_2| & x_1 = x_2 \\ |y_1| + |x_1 - x_2| + |y_2| & x_1 \neq x_2 \end{cases}$$

2.1 Metric Subspaces and Product Spaces

If (X,d) is a metric space, and A is a subset of X, then the restriction $d|_A$, also denoted d_A, of d to A is also a metric on A. Then, we say that (A,d_A) is a (*metric*) *subspace* of of (X,d).

For instance, any set $A \subseteq \mathbb{R}$ equipped with the usual Euclidean metric with the appropriate restriction is a metric subspace of $\mathbb{R}$.

If (X,d_1) and (Y,d_2) are metric spaces, we can define a metric on the Cartesian product of their underlying sets. In fact, there are many ways to do so:

Lemma 2.1. *Let (X,d_1) and (Y,d_2) be metric spaces. Then, for any $1 \leq p \leq \infty$,*

$$\varrho_p\big((x_1,y_1),(x_2,y_2)\big) := \begin{cases} \Big(\big(d_1(x_1,x_2)\big)^p + \big(d_2(y_1,y_2)\big)^p\Big)^{\frac{1}{p}} & 1 \leq p < \infty \\ \max\big(d(x_1,x_2),d(y_1,y_2)\big) & p = \infty \end{cases}$$

defines a metric on $X \times Y$.

That is, we perform a pairwise computation analogous to the ℓ^p metrics on the components of the points. This similarly extends to any finite product of metric spaces.

Theorem 2.2. *Given a finite collection $\big((X_i,d_i)\big)_{i=1}^n$ of metric spaces,*

$$\varrho_p(\mathbf{a},\mathbf{b}) = \begin{cases} \left(\displaystyle\sum_{i=1}^n \big(d_i(a_i,b_i)\big)^p\right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \displaystyle\max_{1 \leq i \leq n} \big(d_i(a_i,b_i)\big) & p = \infty \end{cases}$$

defines a metric on $\prod_{i=1}^n X_i$.

2.2 Open and Closed Sets

Let (X,d) be a metric space. The *open ball* centred at $a \in X$ of radius r is the set,

$$B(a,r) = \{x \in X : d(x,a) < r\}$$

also denoted by $\mathbb{B}_r(a)$.

Similarly, the *closed ball* centred at $a \in X$ of radius r is the set,

$$\overline{B}(a,r) = \{x \in X : d(x,a) \leq r\}$$

also denoted by $\overline{\mathbb{B}}_r(a)$.

Example. The open ball of radius 1 centred at 0 in $\mathbb{R}$ under the Euclidean metric is the interval $\mathbb{B}(0,1) = (-1,1)$. In the subspace $[0,2] \subset \mathbb{R}$, the same ball is instead given by $\mathbb{B}(0,1) = [0,1)$, so balls depend on the ambient space containing them.

Let (X,d) be a metric space. A subset $S \subseteq X$ is *bounded* if there exists $a \in X$ and $r > 0$ such that $S \subset \mathbb{B}(a,r)$.

A subset $U \subseteq X$ is *open in* X if for every $x \in U$ there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U$. A subset $F \subseteq X$ is *closed in* X if its complement $X \setminus F$ is open.

Note that the definition of a closed set here is different from, but equivalent to, the definition in MA259 *Multivariable Calculus.*

Example.

- In any metric space (X,d), X and $\emptyset$ are both simultaneously open and closed (or *clopen*).
- In $\mathbb{R}$, open intervals are open and closed intervals are closed. Half-open intervals are neither open nor closed.
- In a discrete metric space, every singleton set $\{x\} \subseteq X$ is open (take any $\varepsilon < 1$).

Sets can be open, closed, both (clopen), or neither, so the adjectives “open” and “closed” do not have all of their usual intuitive connotations when used in a mathematical context.

Lemma (Open Balls)**.** *Open balls are open sets.*

Proof. Let (X,d) be a metric space, and let $a \in X$ and $r > 0$. Let $y \in \mathbb{B}(a,r)$ so $d(y,a) < r$, and take $\varepsilon := r - d(y,a) > 0$. Then, $\mathbb{B}(y,\varepsilon) \subset \mathbb{B}(a,r)$, since, if $d(z,y) < \varepsilon$, we have,

$$d(z,a) \leq d(z,y) + d(y,a) < \varepsilon + d(y,a) = r$$

■

Corollary (Closed Balls)**.** *Closed balls are closed sets.*

Lemma (Open Finite Intersection)**.** *If $(U_i)_{i=1}^n$ is a finite collection of sets open in (X,d), then $\bigcap_{i=1}^n U_i$ is open in (X,d).*

Proof. Take $x \in \bigcap_{i=1}^n U_i$. Then, for each i, $x \in U_i$, so there exists $\varepsilon_i > 0$ such that $\mathbb{B}(x,\varepsilon_i) \subset U_i$. If $\varepsilon := \min(\varepsilon_1, \dots, \varepsilon_n)$, then,

$$\mathbb{B}(x,\varepsilon) \subseteq \mathbb{B}(x,\varepsilon) \subset U_i$$

for all i, and hence $\mathbb{B}(x,\varepsilon) \subset \bigcap_{i=1}^n U_i$. ■

However, the countable intersection of open sets is not necessarily open. For example, $\left(-\frac{1}{n},\frac{1}{n}\right)$ is open in $\mathbb{R}$ for all n, but,

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n},\frac{1}{n}\right) = \{0\}$$

which is not open in $\mathbb{R}$.

Corollary (Closed Finite Union)**.** *If $(F_i)_{i=1}^n$ is a finite collection of sets closed in (X,d), then $\bigcup_{i=1}^n F_i$ is closed in (X,d).*

Proof. By De Morgan's laws,

$$X \setminus \bigcup_{i=1}^{n} F_i = \bigcap_{i=1}^{n} (X \setminus F_i)$$

As F_i is closed, $X \setminus F_i$ is open, so $\bigcap_{i=1}^{n} (X \setminus F_i)$ is the finite intersection of open sets, and hence $X \setminus \bigcup_{i=1}^{n} F_i$ is open. It follows that $\bigcup_{i=1}^{n} F_i$ is closed. ■

Again, the countable union of closed sets is not necessarily closed. For example, $\left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$ is closed in $\mathbb{R}$ for all n, but,

$$\bigcup_{n=1}^{\infty} \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] = (-1, 1)$$

which is not closed in $\mathbb{R}$.

Lemma (Open Arbitrary Union)**.** *If $(U_i)_{i \in \mathcal{I}}$ is an arbitrary collection of sets open in (X, d), then $\bigcup_{i \in \mathcal{I}} U_i$ is open in (X, d).*

Proof. If $x \in \bigcup_{i \in \mathcal{I}} U_i$, then $x \in U_i$ for some $i \in \mathcal{I}$. Since U_i is open, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U_i \subseteq \bigcup_{i \in \mathcal{I}} U_i$, so $\bigcup_{i \in \mathcal{I}} U_i$ is open. ■

Corollary (Closed Arbitrary Intersection)**.** *If $(F_i)_{i \in \mathcal{I}}$ is an arbitrary collection of sets closed in (X, d), then $\bigcup_{i \in \mathcal{I}} F_i$ is closed in (X, d).*

Proof.

$$X \setminus \bigcap_{i \in \mathcal{I}} F_i = \bigcup_{i \in \mathcal{I}} (X \setminus F_i)$$

and apply the preceding lemma. ■

2.3 Convergence of Sequences

We will now rephrase the ε-δ notion of convergence of sequences from analysis in terms of open sets in metric spaces.

A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) converges to $x \in X$ if,

$$\lim_{n \to \infty} d(x_n, x) = 0$$

or equivalently, in terms of open balls, for every $\varepsilon > 0$, there exists $N \geq 1$ such that,

$$x_n \in \mathbb{B}(x, \varepsilon)$$

for all $n \geq N$.

Lemma 2.3. *A sequence in a metric space can have at most one limit.*

Proof. Suppose that $(x_n) \to a$ and $(x_n) \to b$ so,

$$\lim_{n \to \infty} d(x_n, a) = \lim_{n \to \infty} d(x_n, b) = 0$$

Then,

$$0 \leq d(a, b) \leq d(a, x_n) + d(x_n, b) \to 0$$

so $d(a, b) = 0$ and hence $a = b$. ■

This may be rather familiar from analysis, but it turns out that this result may not hold in more general spaces.

We can now characterise convergence purely in terms of open sets, without directly invoking the metric:

Lemma (Open Set Convergence)**.** *Let $(x_i)_{i=1}^{\infty} \subset (X,d)$ be a sequence. Then, $(x_n) \to x \in X$ if and only if for every open set U containing x, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.*

Lemma (Sequential Closure)**.** *A subset F of a metric space is closed if and only if whenever a sequence $(x_i)_{i=1}^{\infty} \subset F$ converges to some $x \in X$, then $x \in F$.*

3 Continuity

3.1 Metric Continuity

Let $f : (X,d_X) \to (Y,d_Y)$ be a function between two metric spaces. For each point $p \in X$, we write,

$$\lim_{x \to p} f(x) = y \in Y$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < d_X(x,p) < \delta$, we have $d_Y(f(x),y) < \varepsilon$.

Then, f is *continuous at a point* $p \in X$ if $\lim_{x \to p} f(x) = f(p)$. That is, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_X(x,p) < \delta \to d_Y(f(x),f(p)) < \varepsilon$. We also say that f is *continuous on a (sub)set* $S \subseteq X$ if it is continuous at every point $p \in S$.

A function $f : X \to Y$ is *Lipschitz continuous* if there exists a constant $M \geq 0$ such that

$$d_Y(f(x),f(y)) \leq M d_Y(x,y)$$

for every $x,y \in X$, and we say that M is a *Lipschitz constant* or *modulus of (uniform) continuity* for f. Lipschitz continuity implies continuity, as we can take $\delta = \frac{\varepsilon}{M}$.

Let $A \subset X$ be non-empty. We define the distance of a point $x \in X$ from the set A to be,

$$d(x,A) = \inf_{a \in A} d(x,a)$$

Lemma 3.1. *If $A \subset X$ is non-empty, then the function $x \mapsto d(x,A)$ is Lipschitz with modulus* 1.

Proof. Let $x,y \in X$. Then, for every $a \in A$, we have,

$$d(x,A) \leq d(x,a) \leq d(x,y) + d(y,a)$$

Taking the infimum, we have,

$$d(x,A) \leq d(x,y) + d(y,A)$$
$$d(x,A) - d(y,A) \leq d(x,y)$$

The situation is symmetric with respect to y, so we also have,

$$d(y,A) - d(x,A) \leq d(x,y)$$

giving,

$$|d(x,A) - d(y,A)| \leq d(x,y)$$

■

Lemma (Sequential Continuity)**.** *Let (X,d_X) and (Y,d_Y) be metric spaces, and let $(x_i)_{i=1}^{\infty} \subset X$ be a sequence such that $(x_n) \to x \in X$. Then, a function $f : X \to Y$ is continuous at x if and only if $f(x_n) \to f(x)$.*

Lemma (Algebra of Continuous Functions)**.** *Let (X,d) be a metric space. Then,*

- *If $f,g : X \to \mathbb{R}$ are continuous, then $f + g$ and fg are continuous, and f/g is continuous at all points x where $g(x) \neq 0$;*
- *If $(Y,\|\cdot\|)$ is a normed vector space, and $f,g : X \to Y$ are continuous, then $f + g$ is coninuous.*

Continuity and open sets are closely related, but perhaps not in the intuitive way one might expect. In particular, the image of an open set under a continuous function need not be open (or closed). For instance, $\sin : \mathbb{R} \to \mathbb{R}$ sends the open set $(-\pi,\pi)$ to the closed set $[-1,1]$, and the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{1+x^2}$ sends the open set $\mathbb{R}$ to the set $(0,1]$, which is neither open nor closed.

Instead, the *preimage* of any open set under a continuous function is open. If $f : X \to Y$ is a function and $A \subseteq Y$, then we write the preimage of A under f as,

$$f^{-1}(A) = \{x \in X : f(x) \in A\}$$

Note that this does not require that f is invertible.

We now characterise continuity in terms of open sets:

Theorem (Characterisation of Continuity)**.** *For any function $f : X \to Y$, the following statements are equivalent:*

- *f is continuous at all points of X;*
- *$f^{-1}(U)$ is open whenever $U \subseteq Y$ is open;*
- *$f^{-1}(\mathcal{F})$ is closed whenever $\mathcal{F} \subseteq Y$ is closed.*

Note that this does not imply that the image of an open (closed) set under a continuous function is open (resp. closed): only inverse images preserve the topology of a set.

Lemma (Continuity of Compositions)**.** *Suppose that (X,d_X), (Y,d_Y), and (Z,d_Z) are metric spaces, and $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Then, the composition $g \circ f : X \to Z$ is continuous.*

A direct ϵ-δ proof is long and tedious, but using the inverse image characterisation of continuity simplifies the proof considerably:

Proof. If $U \subseteq Z$ is open, then $g^{-1}(U)$ is open in Y, and hence $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in X. ∎

3.2 Topologically Equivalent Metrics

Suppose we have two metrics d_1 and d_2 defined on a set X. We have characterised continuity in terms of open sets, so if the open sets (X,d_1) are the same as the open sets in (X,d_2), then any function $f : X \to Y$ that is continuous on (X,d_1) should be continuous on (X,d_2).

More formally, we note that $f = f \circ \mathrm{id}_X$, so we should require that the identity function $\mathrm{id}_X : X \to X$ is also continuous. But id_X is continuous from (X,d_1) to (X,d_2) if and only if every set that is open in (X,d_2) is also open in (X,d_1).

Lemma 3.2. *Suppose d_1 and d_2 are metrics on X. Then, the following statements are equivalent:*

(i) Every set that is open in (X,d_2) is open in (X,d_1);

(ii) For any metric space (Y,d_Y), if $g : X \to Y$ is continuous as a function $(X,d_2) \to (Y,d_Y)$, then g is continuous as a function $(X,d_1) \to (Y,d_Y)$;

(iii) *For any metric space* (Y,d_Y)*, if* $f : Y \to X$ *is continuous as a function* $(Y,d_Y) \to (X,d_1)$*, then* f *is continuous as a function* $(Y,d_Y) \to (X,d_2)$*.*

Proof. We only show $(i) \leftrightarrow (ii)$, as the proof of $(i) \leftrightarrow (iii)$ is similar.

It follows from (i) that the identity map $\mathrm{id}_X : (X,d_1) \to (X,d_2)$ is continuous. So, if $g : (X,d_2) \to Y$ is continuous, then the composition $g \circ \mathrm{id}_X : (X,d_1) \to Y$ is continuous.

For the reverse implication, take $(Y,d_Y) = (X,d_2)$ and $g = \mathrm{id}_X : (X,d_2) \to (X,d_1)$. Since g is continuous from (X,d_2) to (X,d_2), it is continuous from (X,d_1) to (X,d_2), so for every open set U in (X,d_2), $g^{-1}(U) = U$ is open in (X,d_1). ■

Also applying this lemma in reverse, we obtain,

Theorem 3.3. *Suppose* d_1 *and* d_2 *are metrics on* X*. Then, the following statements are equivalent:*

- *(i) The open sets in* (X,d_2) *and* (X,d_1) *coincide;*
- *(ii) For any metric space* (Y,d_Y)*,* $g : X \to Y$ *is continuous as a function* $(X,d_1) \to (Y,d_Y)$ *if and only if it is continuous as a function* $(X,d_2) \to (Y,d_Y)$*;*
- *(iii) For any metric space* (Y,d_Y)*,* $f : Y \to X$ *is continuous as a function* $(Y,d_Y) \to (X,d_1)$ *if and only if it is continuous as a function* $(Y,d_Y) \to (X,d_2)$*.*

In this case, we say that d_1 and d_2 are *topologically equivalent.*

Two metrics d_1 and d_2 on X are *Lipschitz equivalent* if there exist constants $0 < m \leq M < \infty$ such that,

$$md_1(x,y) \leq d_2(x,y) \leq Md_1(x,y)$$

for all $x,y \in X$.

Lemma 3.4. *If* d_1 *and* d_2 *are Lipschitz equivalent on* X*, then* d_1 *and* d_2 *are topologically equivalent.*

The converse of this lemma does not hold, so Lipschitz equivalence is a strictly stronger notion of equivalence.

Corollary 3.4.1. *Metrics induced by norms are topologically equivalent if and only if the norms are equivalent.*

3.3 Isometries and Homeomorphisms

Suppose $f : X \to Y$ is a bijection such that,

$$d_X(x,y) = d_Y(f(x),f(y))$$

for all $x,y \in X$. That is, f *preserves distances.* Then, f is an *isometry*, and X and Y are *isometric.* Isometric spaces are essentially the same metric spaces, just with different labelling of points, and in fact, isometries are exactly the isomorphisms of metric spaces.

If f and f^{-1} are both additionally continuous (f is *bicontinuous*), then f is a *homeomorphism*, and X and Y are *homeomorphic.* If two spaces are homeomorphic, their open sets coincide, and the spaces are essentially the same *topological* spaces, just with different labelling of points, and in fact, homeomorphisms are exactly the isomorphisms of topological spaces.

Example.

- Every metric space is homeomorphic to itself under the identity map.
- Any two open intervals (a,b) and (α,β) are homeomorphic under

$$f(x) = \alpha + \frac{\beta - \alpha}{b - a}(x - a)$$

- $(-1,1)$ is homeomorphic to $\mathbb{R}$ under

$$f(x) = \tan\left(\frac{\pi x}{2}\right) \qquad \text{or} \qquad f(x) = \frac{x}{1-|x|}$$

- Any open interval is homeomorphic to $\mathbb{R}$ by composing the previous two examples.
- The square is homeomorphic to the circle under a radial projection mapping.

Two metrics d_1 and d_2 on X are topologically equivalent if and only if the identity map $\mathrm{id}_X : (X,d_1) \to (X,d_2)$ is a homeomorphism.

3.4 Topological Properties

If a property P of a metric space is preserved under homeomorphism, then P is a *topological property.* Informally, topological properties are generally those properties that are set-theoretic in nature, and do not care about the exact notion of distance imposed on the space.

Example. Topological properties on a space X:

- X is open in X;
- X is closed in X;
- X is finite; countably infinite; uncountable;
- X has a point x such that $\{x\}$ is open in X (an *isolated point*);
- X has no isolated points;
- Every subset of X is open;
- Every continuous real-valued function on X is bounded.

Example. Non-topological properties (they intrinsically depend on the metric in some way):

- X is bounded;
- For each $r > 0$ there exists a finite set F such that every ball of radius r intersects F (X is *totally bounded*);

4 Topological Spaces

In light of this, it seems that many properties of a space do not depend on our exact choice of measure of distance, and we have already characterised convergence and continuity in metric spaces entirely in terms of open sets. We then may be prompted to dispense with the metric entirely, and define a new kind of space entirely in terms of open sets.

A *topology* on a set T is a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(T)$, which we will call the "*open sets*" of T, such that,

(T1) T and $\emptyset$ are open;

(T2) The intersection of finitely many open sets is open;

(T3) Arbitrary unions of open sets are open.

The pair $(T,\mathcal{T})$ is then a *topological space.*

Example.

- In any metric space (X,d), the induced collection of open sets forms a topology on the underlying set X;
- The *discrete topology* – every set is open (induced by the discrete metric);
- The *indiscrete* or *trivial topology* – only T and $\emptyset$ are open;
- The *cofinite topology* – a set is open if it is T, $\emptyset$, or if its relative complement in T is finite;
- The *cocountable topology* – a set is open if it is T, $\emptyset$, or its relative complement in T is countable;
- The *Zariski topology* on $\mathbb{R}^n$ – a set is open if it is $\mathbb{R}^n$, $\emptyset$, or its complement is the set of zeros of some polynomial $p \in \mathbb{R}[x]$.

Not every topology is induced by a metric. That is, there does not necessarily always exist a metric on T that induces the same collection of open sets as $\mathcal{T}$. More formally, there does not always exist a metric d such that the metric space (T,d) is homeomorphic to the topological space $(T,\mathcal{T})$. If such a metric exists, then $(T,\mathcal{T})$ is *metrisable.*

Theorem 4.1. *The indiscrete topology is not metrisable on any set with more than one point.*

Proof. Suppose the indiscrete topology is on T is induced by some metric d on T. Let $x,y \in T$ be distinct, so $d(x,y) = \varepsilon > 0$. The open ball $\mathbb{B}\left(x,\frac{\varepsilon}{2}\right)$ is open in (T,d). This ball contains x, so it is not the empty set, and it does not contain y, so it is not T. But, $\emptyset$ and T are the only open sets in the indiscrete topology. ∎

It is sometimes possible to compare two topologies on the same space if they are subsets of one another. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on T, such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then we say that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$, or that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$. This defines a partial order on the set of topologies on T.

Sometimes we will say that a topology $\mathcal{T}$ is the coarsest or smallest topology that satisfies a given property P. This means that if $\mathcal{T}'$ also satisfies P, then $\mathcal{T}' \subseteq T$.

The discrete topology is the finest possible topology, and the indiscrete topology is the coarsest possible topology.

The *closed* sets in a topological space are the complements of open sets. By De Morgan's laws, the collection $\mathcal{F}$ of closed sets satisfies:

(F1) T and $\emptyset$ are closed;

(F2) The union of finitely many closed sets is closed;

(F3) Arbitrary intersections of closed sets are closed.

Because the closed sets completely determine the open sets, we can equivalently define a topology in terms of its open sets. In some cases, this is easier then specifying the open sets; for instance, the cofinite topology can be more naturally specified as the topology with finite closed sets.

4.1 Bases

A *basis* for a topology $\mathcal{T}$ on T is a collection $\mathcal{B} \subseteq \mathcal{T}$ such that every set in $\mathcal{T}$ is the union of sets in $\mathcal{B}$.

Example. A set U is open in a metric space (X,d) if for every $x \in U$, there exists $\varepsilon_x > 0$ such that $\mathbb{B}(x,\varepsilon_x) \subset U$, so,

$$U = \bigcup_{x \in U} \mathbb{B}(x,\varepsilon_x)$$

so in any metric space, the collection of all open balls forms a basis for the induced topology.

A topology may have several distinct bases, but each basis generates a unique topology.

Theorem (Uniqueness of Topology for Basis)**.** *If $\mathcal{B}$ is a basis for two topologies $\mathcal{T}$ and $\mathcal{T}'$, then $\mathcal{T} = \mathcal{T}'$.*

Proof. Every set in $\mathcal{T}'$ is a union of sets in $\mathcal{B} \subset \mathcal{T}$, so $\mathcal{T}' \subseteq \mathcal{T}$. The situation is symmetric, so $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T} = \mathcal{T}'$. ■

Theorem (Synthetic Bases)**.** *If $\mathcal{B}$ is a basis for $\mathcal{T}$ on T, then,*

(B1) T is the union of some sets from $\mathcal{B}$;

(B2) If $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2$ is the union of some sets from $\mathcal{B}$.

Conversely, let T be a set and let $\mathcal{B} \subseteq \mathcal{P}(T)$ satisfy (B1) and (B2). Then there is a unique topology $\mathcal{T}$ on T whose basis is $\mathcal{B}$; that is, the open sets are exactly those formed from union of sets from $\mathcal{B}$.

$\mathcal{T}$ is then also the smallest topology that contains $\mathcal{B}$.

Proof. We verify that the set $\mathcal{T}$ generated by the basis is a topology:

(T1) T is the union of sets from $\mathcal{B}$ by (B1);

(T2) If $U, V \in \mathcal{T}$, then $U = \bigcup_{i \in \mathcal{I}} B_i$ and $V = \bigcup_{j \in \mathcal{J}} C_j$ with $B_i, C_i \in \mathcal{B}$. Then,

$$U \cap V = \bigcup_{i \in \mathcal{I}, j \in \mathcal{J}} B_i \cap C_j$$

which is a union of sets in $\mathcal{B}$ by (B2), and is hence an element of $\mathcal{T}$;

(T3) Any union of union of sets from $\mathcal{B}$ is a union of sets from $\mathcal{B}$.

So, $\mathcal{T}$ is a topology on T. ■

A *sub-basis* for a topology $\mathcal{T}$ on a set T is a collection $\mathcal{B} \subseteq T$ such that every set in T is a union *of finite intersections* of sets in $\mathcal{B}$.

Example. One sub-basis of $\mathbb{R}$ with the standard topology is given by,

$$\mathcal{B} = \{(a,\infty), (-\infty, b) : a, b \in \mathbb{R}\}$$

as intersections give the open intervals (a,b) which form a normal basis.

Theorem 4.2. *If $\mathcal{B}$ is any collection of subsets of a set T whose union is T, then there is a unique topology $\mathcal{T}$ on T with sub-basis $\mathcal{B}$, formed precisely from the collection of all unions of finite intersections of sets from $\mathcal{B}$.*

Proof. If $\mathcal{B}$ is a sub-basis for a topology $\mathcal{T}$, then this topology has $\mathcal{D}$, the collection of finite intersections of elements of $\mathcal{B}$ as a basis. But $\mathcal{D}$ satisfies (B1) and (B2), so there is a unique topology $\mathcal{T}$ with basis $\mathcal{D}$ by the previous lemma, which is also the unique topology with sub-basis $\mathcal{B}$. ■

$\mathcal{T}$ is the smallest topology on T that contains $\mathcal{B}$.

4.2 Topological Subspaces and Finite Product Spaces

If $(T,\mathcal{T})$ is a topological space, and $S \subseteq T$, then the *subspace topology* on S is,

$$\mathcal{T}_S = \{U \cap S : U \in \mathcal{T}\}$$

and we call $(S,\mathcal{T}_S)$ a (*topological*) *subspace* of $(T,\mathcal{T})$.

Lemma (Induced Subspaces)**.** *Suppose (X,d) is a metric space with induced topology $\mathcal{T}$. If $S \subseteq X$, then the subspace topology $\mathcal{T}_S$ on S corresponds to the topology on S induced by the metric subspace $(S,d|_S)$.*

If $(T_1,\mathcal{T}_1)$ and $(T_2,\mathcal{T}_2)$ are topological spaces, then the *product topology* on $T_1 \times T_2$ is the topology $\mathcal{T}$ with basis,

$$\mathcal{B} = \{U_1 \times U_2 : (U_1,U_2) \in T_1 \times T_2,\}$$

and we call $(T_1 \times T_2,\mathcal{T})$ the (*topological*) *product* of T_1 and T_2.

Intuitively, the topological product is the smallest topology for which the left and right projections $\pi_1 : T_1 \times T_2 \to T_1$ and $\pi_2 : T_1 \times T_2 \to T_2$ are continuous.

For finite n, the product topology on $\mathbb{R}^n$ agrees with the topologies induced by any of the ϱ_p metrics.

4.3 Closures, Interiors, and Boundaries

Let $(T,\mathcal{T})$ be a topological space. A *neighbourhood* of a point $x \in T$ is a set $N \subseteq T$ that contains an open set $U \in \mathcal{T}$ such that $x \in U \subseteq N$. An *open neighbourhood* of $x \in T$ is an open set $U \in \mathcal{T}$ that contains x. Neighbourhoods are not used as often, so the term sometimes refers to an open neighbourhood.

The *closure* $\overline{A}$ of a set $A \subseteq T$ is the intersection of all closed sets that contain A:

$$\overline{A} = \bigcap_{\substack{A \subseteq \mathcal{F} \\ \mathcal{F} \text{ closed}}} V$$

Note that if A is non-empty, then $\overline{A}$ is non-empty; $\overline{A}$ is also always closed, since it is the intersection of closed sets; the closure of A is therefore the minimal closed set that contains A. It follows that A is closed if and only if $A = \overline{A}$.

For any sets A and B,

- $A \subseteq B \to \overline{A} \subseteq \overline{B}$;
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
- in general, $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

We give an alternative characterisation of closures:

Theorem 4.3 (Characterisation of Closures)**.** *Given $A \subseteq T$, the closure $\overline{A}$ is the set,*

$$\begin{aligned}\overline{A} &:= \{x \in T : U \cap A \neq \emptyset \text{ for all open neighbourhoods } U \in \mathcal{T} \text{ of } x\} \\ &= \{x \in T : \text{every open neighbourhood of } x \text{ intersects } A\}\end{aligned}$$

Theorem (Closure in Metric Spaces)**.** *If (X,d) is a metric space, and $A \subseteq X$, then,*

$$\overline{A} = \{\text{limits of convergent sequences in } A\}$$

The *interior* A° of a set $A \subseteq T$ is the union of all open subsets of A:

$$A^\circ = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$$

Since A° is the union of open sets, it is open, and it is contained in A. It is the maximal open subset of A. So, A is open if and only if $A = A^\circ$.

For any sets A and B,

- $A \subseteq B \to A^\circ \subseteq B^\circ$;
- $(A \cap B)^\circ = A^\circ \cap B^\circ$;
- in general, $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

We give an alternative characterisation of interiors:

Theorem (Characterisation of Interiors)**.** *Given $A \subseteq T$, the interior A° is the set of all points for which A is an open neighbourhood:*

$$A^\circ \coloneqq \{x \in T : x \in U \subseteq A, U \in \mathcal{T}\}$$

Proof. If $x \in A^\circ$, then $x \in U$ for some open set $U \subseteq A$, so A is a neighbourhood of x. Conversely, if A is a neighbourhood of x, then there is an open subset $U \subseteq A$ such that $x \in U$, so $x \in A^\circ$. ∎

Theorem 4.4. *If $A \subseteq T$, then,*

$$A^\circ = T \setminus \overline{T \setminus A} \qquad \textit{and} \qquad \overline{A} = T \setminus (T \setminus A)^\circ$$

The *boundary* ∂A of a set A is the set of all points x such that every neighbourhood of x intersects both A and and its complement.

It is immediate from the definition that

$$\partial A = \overline{A} \cap \overline{T \setminus A}$$

so ∂A is always closed. By the previous theorem, we also have,

$$\begin{aligned} \partial A &= \overline{A} \cap (T \setminus A^\circ) \\ &= \overline{A} \setminus A^\circ \end{aligned}$$

Let $S \subseteq T$. A point $x \in T$ is a *limit point* of S if every neighbourhood of x intersects $S \setminus \{x\}$. Note that a limit point of S need not lie within S. Intuitively, a limit point is "nearby" other points in S, in that, if we remove x from S and look in some neighbourhood around x, then we still see some other points contained in S. In contrast, a point in S that is in S that is not a limit point of S is an *isolated point.*

Example.

- If $S = (0,1) \subset \mathbb{R}$, then every point in $[0,1]$ is a limit point of S;
- If $S = [0,1] \cup \{2\}$, then 2 is not a limit point of S, as we can find a neighbourhood containing 2 that does not intersect S, say, $(1.5,2.5)$, so 2 is an isolated point of S.

Note that if S is closed, then it contains all its limit points, or else we would have a limit point $x \in T \setminus S$, which would be an open set containing x that does not intersect $S = S \setminus \{x\}$, so,

$$\overline{S} = S \cup \{x : x \text{ is a limit point of } S\}$$

A subset $A \subseteq T$ is,

- *dense* in T if $\overline{A} = T$;
- *nowhere dense* in T if $(\overline{A})^\circ = \emptyset$.

Equivalently, a set $A \subseteq T$ is nowhere dense if $T \setminus \overline{A}$ is dense in T since,

$$(\overline{A})^\circ = T \setminus \overline{T \setminus \overline{A}}$$

If A is closed, this reduces to if $T \setminus A$ is dense.

4.4 The Cantor Set

We construct a set that is pathological in many ways and serves as a useful counterexample to many propositions.

The (*middle third*) *Cantor set* is constructed as follows:

(0) Let $C_0 = [0,1] \subset \mathbb{R}$.

(1) Remove the open middle third of this set, leaving,

$$C_1 = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$$

(n) From each of the 2^{n-1} closed intervals from C_{n-1}, remove the open middle third to give a new set that consists of 2^n closed intervals.

Note that C_n consists of 2^n closed intervals, each of length 3^{-n}, so their total length is $\left(\frac{2}{3}\right)^n \to 0$ as $n \to \infty$.

Then, the set,

$$C = \bigcap_{n=0}^{\infty} C_n$$

is the (middle third) Cantor set.

Since each C_n is closed, C is closed as it is the intersection of closed sets. C is also non-empty as it contains the endpoints of every open interval removed, but the interior of C is empty, as the set would otherwise have non-zero length. Since C is closed, it is nowhere dense in [0,1].

We have $\partial C = C$, since $\overline{C} = C$, and $C^\circ = \emptyset$. C also contains no isolated points; for any $\varepsilon > 0$, any point in C was at some point in an interval of length less than $\frac{\varepsilon}{2}$, and the two endpoints of this interval are both in C.

4.5 The Hausdorff Property

Recall that a topological space $(T,\mathcal{T})$ is said to be metrisable if there is a metric d on T such that $\mathcal{T}$ consists of the open sets in (T,d). We will give a necessary condition for a topological space to be metrisable in terms of convergence of sequences.

A sequence $(x_n)_{n=1}^{\infty} \subseteq T$ in a topological space $(T,\mathcal{T})$ converges to $x \in T$ if for every open neighbourhood U of x, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.

Note that this is the same definition of convergence for metric spaces we found earlier. However, in topological spaces, this can lead to some unusual behaviours. Take T to have the indiscrete topology where only T and $\emptyset$ are open. Then, any sequence $(x_n) \subset T$ converges to any point x in T; the only open neighbourhood of x is $U = T$, and $x_n \in T$ for all $n \geq 1$, so $(x_n) \to x$ for all $x \in T$. The problem here is that in the indiscrete topology, distinct points cannot be separated into distinct open sets. This motivates the next definition:

A topological space T is *Hausdorff* if for any distinct $x,y \in T$, there exist disjoint open sets U,V such that $x \in U$ and $y \in V$.

Theorem 4.5. *Metric spaces are Hausdorff.*

Proof. Let (X,d) be a metric space. Take distinct $x,y \in X$, and let $\varepsilon = d(x,y) > 0$. Then, $x \in \mathbb{B}(x,\frac{\varepsilon}{2})$ and $y \in \mathbb{B}(y,\frac{\varepsilon}{2})$, but $\mathbb{B}(x,\frac{\varepsilon}{2}) \cap \mathbb{B}(y,\frac{\varepsilon}{2}) = \emptyset$. ■

Theorem 4.6. *In a Hausdorff space T, any sequence has at most one limit.*

Proof. Suppse $(x_n) \to x$ and $(x_n) \to y$ with $x \neq y$. Then, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$. By the definition of convergence, there exist $N_1 \geq 1$ and $N_2 \geq 1$ such that $x_n \in U$ for all $n \geq N_1$ and $x_n \in V$ for all $n \geq N_2$.

If $n \geq \max(N_1,N_2)$, we must simultaneously have $x_n \in U$ and $x_n \in V$, but $U \cap V = \emptyset$. ■

The converse of this theorem does not hold: there exist non-Hausdorff topologies in which convergent sequences have unique limits.

4.6 Topological Continuity

We previously characterised continuity between metric spaces in terms of open sets. We now reverse this to *define* continuity between topological spaces to be in terms of open sets.

A map $f : T_1 \to T_2$ between two topological spaces $(T_1,\mathcal{T}_2)$ and $(T_2,\mathcal{T}_2)$ is *continuous* if $f^{-1}(U) \subseteq T_1$ is open in T_1 whenever $U \subseteq T_2$ is open in T_2.

Example.

- Any constant map that sends every $x \in T_1$ to some fixed $c \in T_2$ is continuous as $f^{-1}(U) = T_1$ if $c \in U$ and $f^{-1}(U) = \emptyset$ if $c \notin U$.
- The identity map $f : T_1 \to T_1$ is continuous if the domain and codomain have the same topology.
- Continuous maps between metric spaces are also continuous maps between the induced topological spaces.
- Any map with a discrete domain is continuous since every set in the domain is open.

To check that a map is continuous, it suffices to check that it is continuous on a (sub)basis.

Lemma (Continuity of Compositions)**.** *Suppose that $(X,\mathcal{T}_X)$, $(Y,\mathcal{T}_Y)$, and $(Z,\mathcal{T}_Z)$ are topological spaces, and $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Then, the composition $g \circ f : X \to Z$ is continuous.*

Proof. If $U \subseteq Z$ is open, then $g^{-1}(U)$ is open in Y, and hence $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in X. ■

Lemma (Algebra of Continuous Functions)**.** *If $f,g : T \to \mathbb{R}$ are continuous, then $f + g$ and fg are continous, and f/g is continous at all points x where $g(x) \neq 0$.*

For a product $X \times Y$, we define the left and right projections,

$$\pi_1 : X \times Y \to X, \qquad \pi_2 : X \times Y \to Y$$

defined by

$$\pi_1(x,y) = x, \qquad \pi_2(x,y) = y$$

Lemma 4.7. *The projection mappings are continuous.*

Proof. If $U \subseteq X$ is open, then $\pi_1^{-1}(U) = U \times Y$, which is open. A similar argument shows π_2 is continuous. ∎

Theorem (Componentwise Continuity)**.** *A function $f : T \to X \times Y$ with components $f = (f_1,f_2)$ is continuous if and only if its components are continuous.*

Proof. $(f_1,f_2) = (\pi_1 \circ f,\pi_2 \circ f)$ and the result follows from the continuity of the projection mappings and the algebra of continuous functions. ∎

4.7 Homeomorphisms

Recall that a homeomorphism between metric spaces is a bijective and bicontinuous map. The notion of a homeomorphism between topological spaces is essentially the same.

Let $(T_1,\mathcal{T}_1)$ and $(T_2,\mathcal{T}_2)$ be topological spaces. A function $f : T_1 \to T_2$ is a homeomorphism if it is bijective and any of the following equivalent conditions hold:

(i) both f and f^{-1} are continuous;

(ii) $U \subseteq T_2$ is open in T_2 if and only if $f^{-1}(U) \subseteq T_1$ is open in T_1;

(ii) $V \subseteq T_1$ is open in T_1 if and only if $f(V) \subseteq T_2$ is open in T_2.

If a homeomorphism between $(T_1,\mathcal{T}_1)$ and $(T_2,\mathcal{T}_2)$ exists, then we say that $(T_1,\mathcal{T}_1)$ and $(T_2,\mathcal{T}_2)$ are *homeomorphic.*

A property of topological spaces is a *topological invariant* or *topological property* if it is preserved by homeomorphisms.

Example.

- T is finite;
- T is Hausdorff;
- T is metrisable;
- every continuous real-valued function on T is bounded.

To show that two topological spaces are not homeomorphic, we can show that one space has a topological invariant that the other does not. For instance, every continuous real-valued function on [0,1], but not $\mathbb{R}$, is bounded, so [0,1] and $\mathbb{R}$ are not homeomorphic.

5 Compactness

A *cover* of a set A is a collection $\mathcal{U}$ of sets whose union contains A. That is,

$$A \subseteq \bigcup_{U \in \mathcal{U}} U$$

and we say that the elements of $\mathcal{U}$ *cover* A. A *subcover* of a cover $\mathcal{U}$ is a subset of $\mathcal{U}$ whose elements still cover A. A cover is *open* if every element of the cover is open.

Example.

- $\mathcal{U} = \{(n-2,n+2) : n \in \mathbb{Z}\}$ is an (open) cover of $\mathbb{R}$, with one possible subcover given by $S = \{(n-2,n+2) : n \in 2\mathbb{Z}\}$;
- $\mathcal{U} = \{(n,n+1) : n \in \mathbb{Z}\}$ is not a cover of $\mathbb{R}$ since it does not cover the integers.

Note that a subcover is a subset of a cover – we do *not* modify (the size of) sets within the cover. That is, while $S = \{(n-1,n+1) : n \in \mathbb{Z}\}$ covers $\mathbb{R}$, it is *not* considered a subcover of $\mathcal{U} = \{(n-2,n+2) : n \in \mathbb{Z}\}$ because $S \not\subseteq \mathcal{U}$.

A topological space T is *compact* if every open cover of T has a finite subcover.

Example.

- $(0,1)$ is not compact because $\mathcal{U} = \{(0,a) : a \in (0,1)\}$ is an open cover with no finite subcover;
- $\mathbb{R}$ is not compact because $\mathcal{U} = \{(-\infty,a), a \in \mathbb{R}\}$ has no finite subcover.

A subset S of T is compact if every open cover of S by subsets of T has a finite subcover. This is equivalent to S being compact with respect to the subspace topology.

Lemma 5.1. *If $(T,\mathcal{T})$ is a topological space and $S \subseteq T$, then the two notions of compactness above are equivalent.*

Theorem (Heine-Borel)**.** *Any closed interval $[a,b]$ is a compact subset of $\mathbb{R}$ with the standard topology.*

Proof. Let $\mathcal{U}$ be a cover of $[a,b]$ by open subsets of $\mathbb{R}$, and let A denote the set of all points $p \in [a,b]$ such that $[a,p]$ can be covered by a finite subcover S of $\mathcal{U}$. We note that A is non-empty as $[a,a] = \{a\}$ can certainly be covered.

A is bounded above by b, so we can define $c := \sup A \leq b$. Since $a \leq c \leq b$, we must have $c \in U$ for some open set $U \in \mathcal{U}$. Since U is open, there exists $\delta > 0$ such that $(c-\delta,c+\delta) \subseteq U$.

Since $c = \sup A$, there exists at least one point $x \in A$ such that $x \in (c-\delta,c]$. Since $[a,x]$ can be covered by S, and $(c-\delta,c+\delta) \subset U \in \mathcal{C}$, it follows that

$$[a,c+\delta) = [a,x] \cup (c-\delta,c+\delta)$$

also be covered by a finite collection of sets from $\mathcal{U}$ – namely $S \cup \{U\}$.

If $c < b$, then this yields a finite subcover of

$$\left[a, \min(c - \tfrac{\delta}{2},b)\right]$$

which contradicts that $c = \sup A$, so $c = b$, and hence a finite subcover of $\mathcal{U}$ covers $[a,b+\delta) \supset [a,b]$, so $[a,b]$ is compact. ∎

Lemma (Closed in Compact is Compact)**.** *Any closed subset S of a compact space T is compact.*

Proof. Let $\mathcal{U}$ be any cover of S by open subsets of T. Because S is closed, $T \setminus S$ is open, so $\mathcal{U} \cup \{T \setminus S\}$ is an open cover of T.

By the compactness of T, there exists an finite open subcover of this cover. This subcover (with extraneous elements like $T \setminus S$ removed) also covers S, so S is compact. ∎

Lemma (Compact in Hausdorff is Closed)**.** *Any compact subspace S of a Hausdorff space T is closed.*

Proof. Let $a \in T \setminus S$. For each $x \in S$, there exist disjoint open sets $U(x)$ and $V(x)$ containing a and x, respectively. The open sets $\{U(x) : x \in S\}$ form an open cover of S, so there is a finite subcover $\big\{U(x_i)\big\}_{i=1}^{n}$ of S. Then,

$$V_a = \bigcap_{i=1}^{n} V(x_i)$$

is open as it is the finite intersection of open sets; contains a as $V(x)$ contains a for all x by construction; and is disjoint from S by the Hausdorff property, and hence $V_a \subseteq T \setminus S$ for any a. Then,

$$T \setminus S = \bigcup_{a \in T \setminus S} V_a$$

so $T \setminus S$ is the union of open sets and is hence open, so S is closed in T. ∎

Lemma (Compact in Metric is Bounded)**.** *Any compact subspace S of a metric space (X,d) is bounded.*

Proof. Fix any $a \in X$. For any $x \in S$, $x \in \mathbb{B}(a,r)$ for all $r > d(a,x)$, so S is covered by the collection of open balls $\mathcal{U} = \{\mathbb{B}(a,r) : r > 0\}$. By compactness of S, there is a finite subcover $S = \{\mathbb{B}(a,r_i)\}_{i=1}^{n}$, so

$$S \subseteq \bigcup_{i=1}^{n} \mathbb{B}(a,r_i) = \mathbb{B}(a, \max_i r_i)$$

and K is bounded. ∎

Corollary (Compact in $\mathbb{R}$ iff Closed, Bounded)**.** *A subset S of $\mathbb{R}$ with the standard topology is compact if and only if it is closed and bounded.*

Proof. Since $\mathbb{R}$ is a metric space, any compact subset is bounded; since $\mathbb{R}$ is Hausdorff, any compact subset is closed.

For the converse, if $S \subset \mathbb{R}$ is bounded, then there exists $r > 0$ such that $S \subseteq [-r,r]$, which is compact in $\mathbb{R}$. Then, S is a closed subset of a compact set, so S is compact. ∎

Theorem 5.2. *Let $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ be a chain of non-empty closed subsets of a compact space T. Then,*

$$\bigcap_{i=1}^{\infty} F_i \neq \emptyset$$

5.1 Compact Products and Compact Subsets of $\mathbb{R}^n$

Theorem (Tychonov)**.** *The product of any collection of compact spaces is compact with the product topology.*

Theorem (Heine-Borel in $\mathbb{R}^n$)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Let S be a compact subset of $\mathbb{R}^n$. It follows from a previous lemma that S is bounded. Metric spaces are also Hausdorff, so S is closed by a previous lemma.

For the converse, suppose that S is bounded, so there exists $r > 0$ such that $S \subseteq [-r,r]^n$. Since $[-r,r]$ is compact (by Heine-Borel in $\mathbb{R}$), it follows that $[-r,r]^n$ is compact (by Tychonov). If S is closed, then it is a closed subset of a compact space, and is hence compact. ∎

Note that this result does not hold in general metric spaces. For instance, (0,1) is bounded and is closed in itself, but is not compact.

5.2 Continuous Functions on Compact Sets

Theorem (Continuous Image of Compact is Compact)**.** *Let $f : T \to S$ be a continuous function between topological spaces. If T is compact, then $f(T) \subseteq S$ is compact.*

Proof. Suppose $\mathcal{U}$ is an open cover of $f(T)$. Then, $f^{-1}(U)$ is open for all $U \in \mathcal{U}$, and the collection $\{f^{-1}(U) : U \in \mathcal{U}\}$ of these sets covers T. Because T is compact, it has a finite subcover $\{f^{-1}(U_i)\}_{i=1}^n$, and hence $\{U_i\}_{i=1}^n$ is a finite subcover of $f(T)$. ∎

This corollary shows that compactness is a topological property.

Theorem 5.3. *Let $f : T \to S$ be a continuous bijection. If T is compact and S is Hausdorff, then f is a homeomorphism.*

Proof. Let $K \in T$ be closed, and hence compact. Then, $f(K)$ is compact, and, since S is Hausdorff, $f(K)$ is closed. The inverse image of K under f^{-1} (that is, under $(f^{-1})^{-1} = f$) is then closed, so f^{-1} is continuous. ∎

Corollary 5.3.1. *If T is non-empty and compact, then a continuous function $f : T \to \mathbb{R}$ is bounded and attains its bounds.*

5.3 Lebesgue Numbers and Uniform Continuity

Let $\mathcal{U}$ be an open cover of a metric space (X,d). A number $\delta > 0$ is called a *Lebesgue number* for $\mathcal{U}$ if for every $x \in X$, there exists an open set $U \in \mathcal{U}$ such that $\mathbb{B}(x,\delta) \subseteq U$.

In general, open covers do not have a Lebesgue number. For instance, $\mathcal{U} = \{(\frac{x}{2},x) : x \in (0,1)\}$ form an open cover of (0,1), but the covering sets become arbitrary small as $x \to 0$, so no Lebesgue number exists.

Theorem 5.4. *Every open cover $\mathcal{U}$ of a compact metric space (X,d) has a Lebesgue number.*

A map $f : (X,d_X) \to (Y,d_Y)$ between metric spaces is *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x),f(y)) < \varepsilon$ whenever $d_X(x,y) < \delta$ for all $x,y \in X$.

As usual in uniform definitions, δ may depend only on ε and not on x nor y.

Theorem (Compact Continuous is Uniform)**.** *A continuous map from a compact metric space into a metric space is uniformly continuous.*

5.4 Sequential Compactness

A subset K of a metric space (X,d) is *sequentially compact* if every sequence in K has a convergent subsequence whose limit lies in K.

Lemma 5.5. *If K us a sequentially compact subset of a metric space, then any open cover of K has a Lebesgue number.*

Proof. Suppose $\mathcal{U}$ is an open cover of K that does not have a Lebesgue number. Then, for every $\varepsilon > 0$, there exists $x \in K$ such that $\mathbb{B}(x,\varepsilon)$ is not contained in any element of $\mathcal{U}$.

Let $(x_n)_{i=1}^{\infty}$ be a sequence such that $\mathbb{B}(x_n,\frac{1}{n})$ is not contained in any element of $\mathcal{U}$ as above. By sequential compactness, (x_n) has a convergent subsequence $(x_{n_i}) \to x \in K$, and since $\mathcal{U}$ covers K, $x \in U$ for some open set $U \in \mathcal{U}$.

Since $\mathcal{U}$ is open, there exists $\varepsilon > 0$ such that $\mathbb{B}(x,\varepsilon) \subseteq U$. Now, take suficiently large i such that $d(x_{n_i},x) < \frac{\varepsilon}{2}$ and $\frac{1}{n_i} < \frac{\varepsilon}{2}$. Then, $\mathbb{B}(x_{n_i},\frac{1}{n_i}) \subseteq \mathbb{B}(x,\varepsilon) \subseteq U$, contradicting the construction of $(x_n) \supseteq (x_{n_i})$. ∎

Theorem (Sequentially Compact is Compact)**.** *A metric subspace is sequentially compact if and only if it is compact.*

The equivalence of compactness and sequential compactness in metric spaces show that they are also equivalent in normed spaces.

However, there exist closed bounded subsets in general normed spaces that are not compact. For instance, the closed unit ball in ℓ^p is not compact for any $1 \leq p \leq \infty$: consider the sequence of basis vectors $(\mathbf{e}_i)_{i=1}^{\infty}$. This sequence has no convergence subsequence, as any such subsequence would necessarily be Cauchy, but,

$$\|\mathbf{e}_i - \mathbf{e}_j\|_{\ell^p} = \begin{cases} 2^{\frac{1}{p}} & 1 \leq p < \infty \\ 1 & p = \infty \end{cases}$$

In fact, the compactness of the closed unit ball is necessary and sufficient for a normed space to be finite-dimensional:

Theorem 5.6. *A normed space is finite-dimensional if and only if its closed unit ball is compact.*

6 Connectedness

A pair of sets (A,B) is a *partition* of a topological space T if $T = A \cup B$ and $A \cap B = \emptyset$, and we say that A and B *partition* T.

It is clear from the definition that if A and B partition T, they are open if and only if they are closed.

A topological space T is *connected* if the only partitions of T into open (closed) sets are $(T,\emptyset)$ and $(\emptyset,T)$, and is *disconnected* otherwise.

Lemma (Characterisation of Disconnected Spaces)**.** *The following statements are all equivalent:*

(i) T is disconnected;

(ii) T has a partition into two non-empty open sets;

(iii) T has a partition into two non-empty closed sets;

(iv) T has a clopen subset equal to neither $\emptyset$ nor T;

(v) there is a continuous function from T to the two-point set $\{0,1\}$ with the discrete topology.

Proof.

$(i) \leftrightarrow (ii)$: Follows from the definition of disconnected.

$(ii) \leftrightarrow (iii)$: If A and B are open and $T = A \cup B$, then $A = T \setminus B$ and $B = T \setminus A$ are closed in T.

$(ii) \leftrightarrow (iv)$: As above, if A,B satisfy the hypotheses of (ii), they are clopen, so (iv) holds. Now, suppose (iv) holds, and let $A \subset T$ be non-empty and clopen. Then, $B = T \setminus A$ is open and A,B partition T.

$(ii) \leftrightarrow (v)$: Let χ_A be the indicator function of A. Then, $\chi_A^{-1}(1) = A$ and $\chi_A^{-1}(1) = B$ and $\chi_A^{-1}(\{0,1\}) = T$, which are all open, so f is continuous. For the converse, suppose $f : T \to \{0,1\}$ is a continuous surjection. Define $A \coloneqq f^{-1}(0)$ and $B \coloneqq f^{-1}(1)$. Both A and B are open as f is continuous; are non-empty as f is surjective; and $A \cup B = T$ and $A \cap B = \emptyset$, so A and B partition T. ■

Statement (v) allows us to show that a space is connected by showing that any continuous function $T \to \{0,1\}$ must be a constant function. Equivalently, we can show that a space is disconnected by exhibiting a non-constant (or equivalently, surjective) function $T \to \{0,1\}$.

Statement (iv) shows that if a subset of T is clopen, then it is either empty or is all of T.

A subset S of T is connected (disconnected) if $(S,\mathcal{T}_S)$ is connected (disconnected). That is, S is connected (disconnected) as a topological space under the subspace topology.

A subset $S \subseteq T$ is *separated* by sets $U,V \in \mathcal{T}$ if,

- $S \subseteq U \cup V$;
- $S \cap U \neq \emptyset$;
- $S \cap V \neq \emptyset$;
- $S \cap U \cap V = \emptyset$.

That is, S is at least (partially) contained within both U and V individually, contained entirely within U and V together, but is not contained in any overlap between U and V (if any such overlap exists).

Theorem 6.1. *A subspace $(S,\mathcal{T}_S)$ of a space $(T,\mathcal{T})$ is disconnected if and only if it is separated by some sets $U,V \in \mathcal{T}$.*

Proof. If S is disconnected, then there are non-empty $A,B \in \mathcal{T}_S$ such that $S = A \cup B$ and $A \cap B = \emptyset$. By the definition of the subspace topology, there exist $U,V \in \mathcal{T}$ such that $A = U \cap S$ and $B = V \cap S$. Then, U and V separate S. Conversely, if U and V separate S, then $U \cap S$ and $V \cap S$ partition S and S is disconnected. ■

6.1 Connected Subsets of $\mathbb{R}^n$

An *interval* of the real line is any set of the form,

- $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$;
- $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$;
- $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$;
- $(a,b) = \{x \in \mathbb{R} : a < x < b\}$.

where $a,b \in \mathbb{R} \cup \{-\infty, +\infty\}$, with infinite values allowed only with strict inequalities.

Lemma 6.2. *A set $I \subseteq \mathbb{R}$ is an interval if and only if whenever $x,z \in I$ and $x < y < z$, then $z \in I$.*

That is to say, an interval contains all points between any pair of points in the interval.

Theorem (Intervals are Connected)**.** *A subset of $\mathbb{R}$ is connected if and only if it is an interval.*

Theorem (Union of Overlapping Connected Sets). *If $(C_i)_{i\in\mathcal{I}}$ are connected subsets of T and $C_i \cap C_j \neq \emptyset$ for all $i,j \in \mathcal{I}$, then,*

$$K = \bigcup_{i\in\mathcal{I}} C_i$$

is connected.

Proof. Suppose $f : K \to \{0,1\}$ is continous. Since each C_i is connected, $f(C_i) = \{\delta_i\}$ where δ_i is either 0 or 1 for each i. Since $C_i \cap C_j$ is non-empty for all $i,j \in \mathcal{I}$, it follows that $f(C_i)$ takes the same value for every $i \in I$, so f must be a constant function and hence K is connected. ■

Lemma 6.3. *Suppose C and D are connected subsets of T and $\overline{C} \cap D \neq \emptyset$. Then, $C \cup D$ is connected.*

Proof. Let $K = C \cup D$, and suppose $f : K \to \{0,1\}$ is continous (and $\{\}$). Suppose $f(C) = \{0\}$ and $f(D) = \{1\}$ (we just require that C and D have disjoint images in $\{0,1\}$).

$\{1\}$ is open in $\{0,1\}$, so $f^{-1}(\{1\})$ is open in K, and is hence $f^{-1}(\{1\}) = U \cap K$ for some open set $U \in T$ by the definition of the subspace topology.

Since $\overline{C} \cap D \neq \emptyset$, there exists $x \in D$ such that every open neighbourhood of x in T intersects C. $U \ni x$ is such a set, so $U \cap C \neq \emptyset$. But, $C \subseteq K$, so this is the same as,

$$\begin{aligned}\emptyset &\neq U \cap C \\ &= U \cap (K \cap C) \\ &= f^{-1}(\{1\}) \cap C\end{aligned}$$

but $f(C) = \{0\}$, so this implies $\{1\} \cap \{0\} \neq \emptyset$. It follows that C and D cannot have disjoint images in $\{0,1\}$, so f must be constant on K, so K is connected. ■

Theorem (Union of Connected Subspaces). *Suppose C and $(C_i)_{i\in\mathcal{I}}$ are connected subspaces of T and $\overline{C} \cap C_i \neq \emptyset$ for all $i \in \mathcal{I}$. Then,*

$$C \cup \bigcup_{i\in\mathcal{I}} C_i$$

is connected.

Proof. Define $C_i' := C \cup C_i$ for $i \in \mathcal{I}$. Then, each C_i' is connected by the previous lemma. We also have $C_i' \cap C_j' = (C \cup C_i) \cap (C \cup C_j) = C \cup (C_i \cap C_j)$ so $C_i' \cap C_j' \neq \emptyset$ for all $i,j \in \mathcal{I}$, and $\bigcup_{i\in\mathcal{I}} C_i' = \bigcup_{i\in\mathcal{I}} C \cup C_i = C \cup \bigcup_{i\in\mathcal{I}} C_i = K$, so the C_i' and K satisfy the hypotheses of the previous theorem, and hence K is connected. ■

Corollary (Subsets of Closure). *If $C \subseteq T$ is connected, then so is any set K satisfying $C \subseteq K \subseteq \overline{C}$.*

Proof. $K = C \cup \bigcup_{x\in K}\{x\}$, and $\{x\} \cap \overline{C} \neq \emptyset$ for all $x \in K$. ■

Theorem (Connected Product). *If topological spaces T and S are connected, then the topological product $T \times S$ is connected.*

Proof. Let $t \in T$, $s \in S$ and define $C \coloneqq T \times \{s_0\}$ and $C_t \coloneqq \{t\} \times S$. Then, C is homeomorphic to T and C_t is homeomorphic to S, so both are connected. $C \cap C_t$ is non-empty as $(t,s) \in C, C_t$, so $C \cap \overline{C_t} \supset C \cap C_t$ is non-empty. We also have,

$$T \times S = C \cup \bigcup_{t \in T} C_t$$

so $T \times S$ is connected by the previous theorem. ■

Theorem (Continuous Image of Connected is Connected). *Let $f : T \to S$ be a continuous function between topological spaces. If T is connected, then $f(T) \subseteq S$ is connected.*

Proof. Suppose T is connected. If $f(T)$ is disconnected, then there exists a non-constant $g : f(T) \to \{0,1\}$. But then, $g \circ f$ is a continuous non-constant function $T \to \{0,1\}$, contradicting that T is connected. It follows that no such g exists, so $f(T)$ is connected. ■

This corollary shows that connectedness is a topological property.

To show that a set is connected, we can show that it can be constructed from the continuous images of known connected sets (often intervals), via products, and via unions.

Example.

- $\mathbb{R}^2$ is connected as $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and $\mathbb{R} = (-\infty,\infty)$ is connected as it is an interval.
- The circle S^1 is connected, as it is the continuous image of $[0,2\pi]$ by $t \mapsto (\sin t, \cos t)$.
- The unit square is connected, as it is the image of S^1 under a radial projection mapping, which is continuous.
- $\mathbb{R}^2 \setminus \{(0,0)\}$ is connected, since $\mathbb{R}^n \setminus \{0\} = \bigcup_{r \in \mathbb{R}} rS^1$ is the union of circles.

We have shown that connectedness is a topological invariant. More useful, however, is that the property "$T \setminus \{x\}$ is connected for every $x \in T$" is a topological property. That is, if $f : T \to S$ is a homeomorphism, then for any $y \in S$, the set $S \setminus \{y\}$ is the continuous image of $T \setminus \{x\}$ for some $x \in T$. This can be used to show that certain sets are not homeomorphic by finding a point that disconnect one set when removed, while proving no such points exist for the other.

Example.

- $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$: $\mathbb{R} \setminus \{0\}$ is disconnected, but $\mathbb{R}^2 \setminus \{(0,0)\}$ is still connected.
- $[0,1]$ is not homeomorphic to S^1: removing an interior point from $[0,1]$ disconnects the set, but removing a point from the circle leaves it connected.
- Similarly, $[0,1]$ is not homeomorphic to the unit square.

6.2 Connected Components

We can define an equivalence relation on a topological space T by having $x \sim y$ if and only if $x,y \in C$ for some connected $C \subseteq T$. This relation is clearly reflexive and symmetric. For transitivity, suppose $x \sim y$ and $y \sim z$, so $x,y \in C_1$ and $y,z \in C_2$. $C_1 \cap C_2$, is non-empty as it contains y, so $C_1 \cup C_2$ is connected and $x,z \in C_1 \cup C_2$, so $x \sim z$.

The equivalence classes of $\sim$ are called the *connected components* of T.

- The connected component C containing x is the union of all connected subsets of T that contain x:

$$C = \bigcup_{x \in S \subseteq T} S$$

- Connected components are connected;
- Connected components are closed;
- Connected components are maximal connected subsets of T.

Example. The connected components of $T = (0,1) \cup (1,2)$ are $(0,1)$ and $(1,2)$. The connected components of $\mathbb{Q}$ are the singleton sets $\{p\}$, where $p \in \mathbb{Q}$.

Since the continuous image of a connected space is connected, the number of connected components is a topological invariant.

6.3 Path-Connected Spaces

Given two points s and t in a topological space T, a *path* from s to t, or a *s-t-path*, is a continuous map $\varphi : [0,1] \to T$ such that $\varphi(0) = s$ and $\varphi(1) = t$.

A space T is *path-connected* if every pair of points T can be joined by a path in T.

Theorem (Path-Connected is Connected)**.** *A path-connected space is connected.*

Proof. Fix $s \in T$, and let $t \in T$. The path $C_v = \varphi([0,1])$ is then connected as it is the continuous image of a connected space. Then, $T = \{u\} \cup \bigcup_{v \in T} C_v$, and each C_v contains u, so T is connected. ■

In general, the converse of this theorem does not hold, so path-connectedness is a stronger notion of connectedness. However, there are some specific cases where the two are equivalent.

Theorem 6.4. *Connected open subsets of $\mathbb{R}^n$ are path-connected.*

Theorem 6.5. *Connected components of open subsets of $\mathbb{R}^n$ are open.*

Proof. Let $U \subseteq \mathbb{R}^n$ be open and let C be one of its connected components. If $x \in C \subseteq U$, then there exists $\varepsilon > 0$ such that $\mathbb{B}(x,\varepsilon) \subseteq U$ as U is open. But C is the union of all connected subsets of U that contain x, so $\mathbb{B}(x,\varepsilon) \subseteq C$ and C is open. ■

Theorem 6.6. *A subset U of $\mathbb{R}$ is open if and only if it is the union of countably many disjoint open intervals:*

$$U = \bigcup_{i \in \mathcal{I}} (a_i,b_i), \qquad (a_i,b_i) \cap (a_j,b_j) = \emptyset \textit{ for all } i \neq j$$

Proof. Any union of open sets is open. For the converse, let $U \subseteq \mathbb{R}$ be open, and let $(C_i)_{i \in \mathcal{I}}$ be the collection of its connected components, which are mutually disjoint. These components are open by the previous theorem, and since they are open and connected, they are open intervals. Then, for each C_i, we can pick a rational q_i in C_i, so we can index the connected components by $\mathbb{Q}$, which is countable. ■

7 Completeness in Metric Spaces

Recall that a sequence $(x_n)_{n=1}^{\infty}$ converges in a metric space (X,d) if and only if it is Cauchy. That is, if for every $\varepsilon > 0$, there exists N such that, for all $n,m \geq N$,

$$d(x_n,x_m) < \varepsilon$$

A metric space (X,d) is *complete* if every Cauchy sequence in X converges, and is *incomplete* otherwise.

It is implicit in the definition that the limit must lie in X. So, for example, $\mathbb{R}$ and $\mathbb{C}$ are complete, but $(0,1)$ is not complete, as $(\frac{1}{n})_{n=1}^{\infty}$ is Cauchy, but $\frac{1}{n} \to 0 \notin (0,1)$. Since $\mathbb{R}$ and $(0,1)$ are homeomorphic (given by $x \mapsto \frac{1}{1+e^{-x}}$, or any other sigmoid curve), this shows that completeness is *not* a topological property.

Theorem 7.1 (Complete is Closed)**.** *Let (X,d) be a metric space, and let $S \subseteq X$. If $(S,d|_S)$ is complete, then S is closed in X.*

Theorem 7.2 (Closed in Complete is Complete)**.** *Let (X,d) be a metric space, and let $S \subseteq X$. If (X,d) is complete and S is closed, then $(S,d|_S)$ is complete.*

Theorem (Compact Metric is Complete)**.** *Any compact metric space (X,d) is complete.*

A normed space is complete if the induced metric space is complete, and is incomplete otherwise. A complete normed space is called a *Banach* space.

Theorem 7.3. *$\mathbb{R}^n$ is complete as a normed space for all $n \in \mathbb{N}$.*

Because all norms are equivalent on $\mathbb{R}^n$, $\mathbb{R}^n$ is complete under all norms.

Theorem 7.4. *ℓ^p is complete for all $1 \leq p \leq \infty$*

Theorem 7.5. *For any non-empty set X, the space $B(X)$ of bounded real-valued functions defined on X, $f : X \to \mathbb{R}$ under the supremum norm,*

$$\|f\|_\infty \coloneqq \sup_{x \in X} |f(x)|$$

is complete.

Theorem 7.6. *For any non-empty topological space $(T,\mathcal{T})$, the space $C_b(T)$ of bounded and continuous real-valued functions defined on X, $f : X \to \mathbb{R}$ under the supremum norm is complete.*

Proof. $C_b(T)$ is a closed subspace of $B(T)$ and is hence complete. ∎

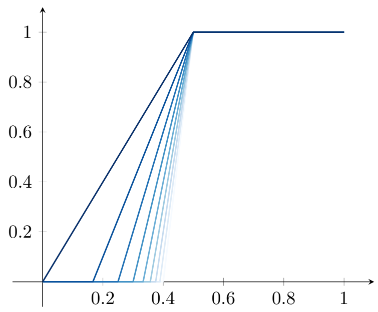
Theorem 7.7. *For any non-empty compact topological space $(T,\mathcal{T})$, the space $C(T)$ of continuous real-valued functions defined on X, $f : X \to \mathbb{R}$ under the supremum norm is complete.*

Proof. If $f \in C(T)$ and T is compact, then f is bounded, so $C(T) = C_b(T)$ and is hence bounded. ∎

7.1 Completions

Consider the space $C[0,1]$ of continuous functions defined on $[0,1]$ under the L^1 norm. This space is not complete as there exist Cauchy sequences that do not converge to a function in $C[0,1]$. For example,

$$f_n(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} - \frac{1}{n} \\ 1 - n(\frac{1}{2} - x) & \frac{1}{2} - \frac{1}{n} \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} < x \leq 1 \end{cases}$$



$f_n(x)$ for $n = 2,3,\ldots$

For $n > m$,

$$\begin{aligned}\|f_n - f_m\|_{L^1} &= \left(\int_0^1 |f_n(x) - f_m(x)|^1 \, dx\right)^{\frac{1}{1}} \\ &= \int_0^1 |f_n(x) - f_m(x)| \, dx \\ &\leq \frac{1}{m}\end{aligned}$$

so this sequence is Cauchy. f converges in the L^1 norm to the function

$$f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

since

$$\begin{aligned}\|f_n - f\|_{L^1} &= \int_0^1 |f_n(x) - f(x)| \, dx \\ &= \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} |f_n(x)| \\ &\leq \frac{1}{n}\end{aligned}$$

Clearly, $f \notin C[0,1]$, so $C[0,1]$ with the L^1 norm is incomplete.

An incomplete space can be *completed* by adding in the missing limit points, resulting in the *completion* of that space. There are two methods of completing an incomplete space A:

(i) Find a complete metric space $X \supset A$ such that A is dense in X. That is, $\overline{A} = X$.

(ii) Find a complete metric space X and an isometry $\mathfrak{i} : A \to Y$ with $Y \subseteq X$ and $\overline{Y} = X$.

Example.

(i) $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{R}$ is complete, so $\mathbb{R}$ is the completion of $\mathbb{Q}$.

(ii) $\mathbb{R}$ is the completion of $\mathbb{Q}$ with isometry $\mathfrak{i}(x) = x$ or $\mathfrak{i}(x) = -x$.

The second method is more flexible in that we do not have to find a complete space that contains exactly A, but only a space isometric to A.

Theorem 7.8. *Every metric space (X,d) can be isometrically embedded into the complete space metric space $B(X)$.*

Corollary 7.8.1. *Every metric space has a completion.*